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COMMENT

Lacunarity and universality

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Abstract. Lacunarity as a criterion of universality in phase transitions on Sierpinski carpets is studied. The approximate theoretical and numerical results suggest that systems of different lacunarity may belong to a common universality class. There seems to be a lack of solutions to the complete set of universality criteria on Sierpinski carpets.

The concept of universality is an essential and important one in phase transitions. It has been recognised that universality depends on a number of parameters, for example Euclidean dimensionality and intrinsic symmetry, and physical systems can be classified accordingly. Renormalisation group theory gives a natural interpretation of universality (for example, Ma 1976). However, for fractal lattices, what a complete set of universality is has still to be fully studied. Recently, Gefen *et al* (1984) have generalised the concept of universality for Sierpinski carpets and found that critical exponents depend on fractal dimension, D, the connectivity, Q, and lacunarity, L.

In this comment we will report a number of results associated with universality on Sierpinski carpets. Our results seem to provide some evidence to show that systems of different lacunarity do not necessarily belong to distinct universality classes, a feature not apparent from the work of Gefen *et al* (1984).

We start with a study of phase transitions of the Potts model on Sierpinski carpets. An approximate bond-moving Migdal-Kadanoff real space renormalisation group technique (Migdal 1975, Kadanoff 1976, Gefen *et al* 1984) yields two basic recursion relations (Lin 1986)

$$e^{-K_i'}=\frac{[1+(q-1)\,e^{-K_m}]^{b-l}[1+(q-1)\,e^{-\tilde{K}}]^l-(1-e^{-K_m})^{b-l}(1-e^{-\tilde{K}})^l}{[1+(q-1)\,e^{-K_m}]^{b-l}[1+(q-1)\,e^{-\tilde{K}}]^l+(q-1)(1-e^{-K_m})^{b-l}(1-e^{-\tilde{K}})^l}\qquad i=1,2 \tag{1}$$

where $K_1'=K'$ and $K_2'=K_w'$ are renormalised coupling parameters between nearest-neighbour spins, b and l are the structure parameters of the Sierpinski carpets (Gefen *et al* 1984) ($l\times l$ subsquares are eliminated from $b\times b$ subsquares) and q is the number of states of the Potts model. The variables K_m and $\tilde{K}$ are defined as

$$\left.\begin{aligned}K_m&=f_1(b,l)K+g_1(b,l)K_w\\ \tilde{K}&=f_2(b,l)K+g_2(b,l)K_w\end{aligned}\right\}\qquad \text{for } K' \tag{2}$$

$$\left.\begin{aligned}K_m&=u_1(b,l)K+v_1(b,l)K_w\\ \tilde{K}&=u_2(b,l)K+v_2(b,l)K_w\end{aligned}\right\}\qquad \text{for } K_w' \tag{3}$$

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in which K and K_w are two types of interaction parameter (Gefen *et al* 1984). The functions f_i, g_i, u_i and v_i ($i=1,2$) depend on the way in which eliminated subsquares are distributed on Sierpinski carpets at fixed b and l. In general, it is thus expected that there exist different recursion relations, fixed points and critical exponents, when the distribution of eliminated subsquares is different, which correspond to different lacunarity, L, for given b and l. For the carpets shown in figure 1, equations (2) and (3) may be written, for the central cutout (see figure 1(a)), as

$$\left.\begin{aligned} K_m &= bK \\ \tilde{K} &= (b-l-1)K+2K_w \end{aligned}\right\} \quad \text{for } K' \tag{4}$$

$$\left.\begin{aligned} K_m &= \tfrac{1}{2}(b-1)K+K_w \\ K &= \tfrac{1}{2}(b-l-2)K+2K_w \end{aligned}\right\} \quad \text{for } K'_w \tag{5}$$

and for the scattered cutout (see figure 1(b)) as

$$\left.\begin{aligned} K_m &= bK \\ \tilde{K} &= K+(b-1)K_w \end{aligned}\right\} \quad \text{for } K' \tag{6}$$

$$\left.\begin{aligned} K_m &= lK+K_w \\ \tilde{K} &= (l+1)K_w \end{aligned}\right\} \quad \text{for } K'_w. \tag{7}$$

The other group of Sierpinski carpets is presented in figure 2. The numerical results for the lacunarity and critical exponents of both groups are summarised in table 1.

However, we found that some Sierpinski carpets with different scattered cutouts, at fixed b and l (i.e. given fractal dimension and connectivity), have the same recursion relations, fixed points and critical exponents, in contrast to the results mentioned above. Figures 3 and 4 give two examples in this respect. Their expressions, (2) and (3), take the forms

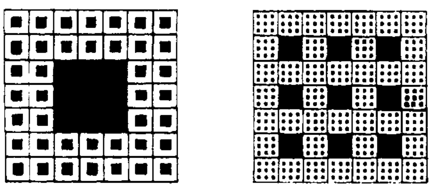
$$\left.\begin{aligned} K_m &= 19K \\ \tilde{K} &= 3K+14K_w \end{aligned}\right\} \quad \text{for } K' \tag{8}$$

and

$$\left.\begin{aligned} K_m &= 9K+K_w \\ \tilde{K} &= K+8K_w \end{aligned}\right\} \quad \text{for } K'_w \tag{9}$$

for figure 3 and

$$\left.\begin{aligned} K_m &= 19K \\ \tilde{K} &= 7K+6K_w \end{aligned}\right\} \quad \text{for } K' \tag{10}$$



(a) (b)

Figure 1. Two-stages of the Sierpinski carpet with $b=7$, $l=3$, fractal dimension $D=1.896$, connectivity $Q=0.712$. (a), central cutout; (b), scattered cutout.

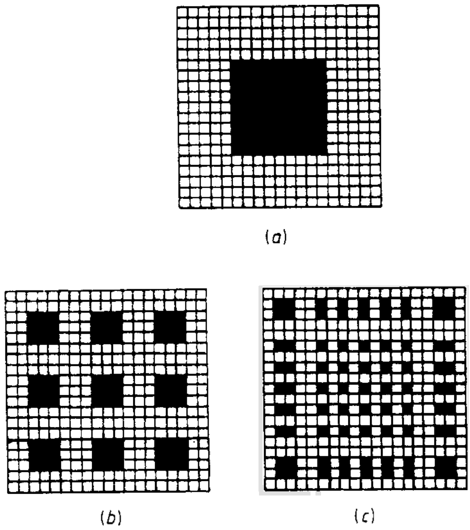


Figure 2. Three types of one-stage of the Sierpinski carpet with $b = 19$, $l = 9$, fractal dimension $D = 1.913$ and connectivity $Q = 0.782$.

Table 1. Results for critical exponents for the Potts model on Sierpinski carpets.

	b	l	L_G	L	q	ν
	7	3	3.942	0.105 57	2	1.908
Figure 1(*a*)					3	1.738
					4	1.701
					5	1.645
	7	3	0.998	0.044 45	2	2.174
Figure 1(*b*)					3	1.969
					4	1.845
					5	1.761
	19	9	208.2	0.092 15	2	1.786
Figure 2(*a*)					3	1.661
					4	1.585
					5	1.558
	19	9	34.00	0.035 45	2	1.969
Figure 2(*b*)					3	1.821
					4	1.730
					5	1.664
	19	9	7.281	0.024 24	2	2.739
Figure 2(*c*)					3	2.632
					4	2.500
					5	2.398

† According to Gefen *et al* (1984), the approximate expression for lacunarity is written as $L_G = (1/n)\,\Sigma_i\,(n_i - \bar{n})^2$ and the approximate expression for lacunarity according to Lin (1986) is $L = (1/N)\,\Sigma^{b}_{=l}\,(1/\bar{n})((1/n)\,\Sigma_i\,(n_i - \bar{n})^2)^{1/2}$.

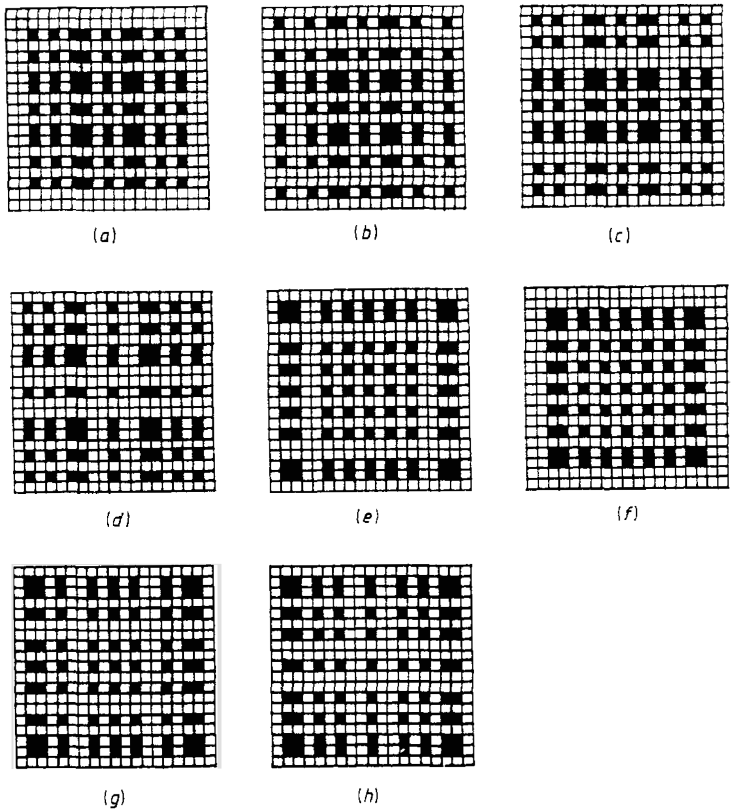


Figure 3. Eight types of one-stage of the Sierpinski carpet with $b=19$, $l=9$, fractal dimension $D=1.913$ and connectivity $Q=0.782$.

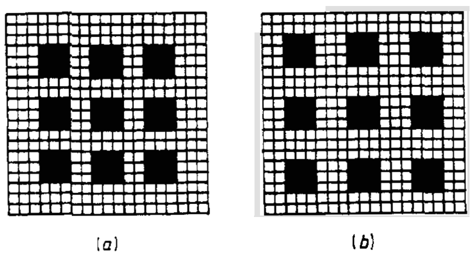


Figure 4. Two types of one-stage of the Sierpinski carpet with $b=19$, $l=9$, fractal dimension $D=1.913$ and connectivity $Q=0.782$.

and

$$\left.\begin{aligned} K_m &= 9K + K_w \\ \tilde{K} &= 3K + 4K_w \end{aligned}\right\} \qquad \text{for } K'_w \tag{11}$$

for figure 4. It is easy to see that there is a common structural feature in both groups in figures 3 and 4. For example, in each structure of figure 3, four 4×4 subsquares, twenty 2×1 subsquares and twenty-five 1×1 subsquares are eliminated. This feature makes it possible to employ 'bond-interchanging invariance' (Lin 1986) and yield the same recursion relations. In table 2 we list their lacunarity, L, and correlation critical

Table 2. Results for the lacunarity and critical exponents on Sierpinski carpets.

(*a*) Results for the lacunarity.

	Figure				
	3(*a*)	3(*b*)	3(*c*)	3(*d*)	3(*e*)
L_G	18.31	24.05	10.00	7.281	7.281
L	0.035 56	0.029 50	0.023 89	0.021 25	0.024 24
	3(*f*)	3(*g*)	3(*h*)	4(*a*)	4(*b*)
L_G	10.00	8.922	13.84	27.78	34.00
L	0.029 53	0.024 65	0.022 86	0.043 77	0.035 45

(*b*) Results for critical exponents.

	b	l	q	ν
	19	9	2	1.969
Figure 3			3	1.821
			4	1.730
			5	1.664
	19	9	2	2.793
Figure 4			3	2.632
			4	2.500
			5	2.398

exponent, ν. The numerical results show that although, for example all structures in figure 3 have different lacunarity, they have the same critical exponents and thus belong to the same universality class.

From all this there seems to be a reason to say that the lacunarity cannot serve as a criterion of universality. However, we have to point out that, since our results are based on an approximate RG scheme and an approximate expression of lacunarity, there still seems to be a lack of conclusive evidence to verify our statements. A full and rigorous study is still necessary. We believe that either more details of structure have to be known to enumerate universality classes, or the validity of universality on Sierpinski carpets fails.

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